FORBIDDEN SUBGRAPHS IN THE NORM GRAPH

SIMEON BALL AND VALENTINA PEPE

Abstract. We show that the norm graph constructed in [11] with n vertices about $\frac{1}{2}n^{2-1/t}$ edges, which contains no copy of $K_{t,(t-1)!+1}$, does not contain a copy of $K_{t+1,(t-1)!-1}$.

1. Introduction

Let H be a fixed graph. The *Turán number* of H, denoted $ex(n,H)$, is the maximum number of edges a graph with n vertices can have, which contains no copy of H. The Erdős-Stone theorem from [7] gives an asymptotic formula for the Turán number of any non-bipartite graph, and this formula depends on the chromatic number of the graph H.

When H is a complete bipartite graph, determining the Turán number is related to the "Zarankiewicz problem" (see [3], Chap. VI, Sect.2, and [9] for more details and references). In many cases even the question of determining the right order of magnitude for $ex(n,H)$ is not known.

Let $K_{t,s}$ denote the complete bipartite graph with t vertices in one class and s vertices in the other. Kővari, Sós and Turán [12] proved that for $s \geqslant t$

$$ex(n,K_{t,s}) \leqslant \tfrac{1}{2}(s-1)^{1/t}n^{2-1/t} + \tfrac{1}{2}(t-1)n. \tag{1.1}$$

The norm graph $\Gamma(t)$, which we will define the next section, has n vertices and about $\frac{1}{2}n^{2-1/t}$ edges. In [1] (based on results from [11]) it was proven that the graph $\Gamma(t)$ contains no copy of $K_{t,(t-1)!+1}$, thus proving that for $s \geqslant (t-1)!+1$,

$$ex(n,K_{t,s}) > cn^{2-1/t}$$

for some constant c.

In [2], it was shown that $\Gamma(4)$ contains no copy of $K_{5,5}$, which improves on the probabilistic lower bound of Erdős and Spencer [6] for $ex(n,K_{5,5})$. In this article, we will generalise this result and prove that $\Gamma(t)$ contains no copy of $K_{t+1,(t-1)!-1}$. For $t \geqslant 5$, this does not

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improve the probabilistic lower bound of Erdős and Spencer,

$$ex(n, K_{t,s}) \geqslant cn^{2-(s+t-2)/(st-1)}.$$

As far as we are aware, it is however the deterministic construction of a graph with n vertices containing no $K_{t+1,(t-1)!-1}$ with the most edges.

2. The norm graph

Suppose that $q = p^h$, where p is a prime, and denote by $\mathbb{F}_q$ the finite field with q elements. We will use the following properties of finite fields. For any $a, b \in \mathbb{F}_q$, $(a+b)^{p^i} = a^{p^i} + b^{p^i}$, for any $i \in \mathbb{N}$. Note that $(a-b)^{p^i} = a^{p^i} - b^{p^i}$, since either p^i is odd or $-1 = 1$. Secondly, for all $a \in \mathbb{F}_{q^i}$, $a^q = a$ if and only if $a \in \mathbb{F}_q$. Finally $N(a) = a^{1+q+\cdots+q^{k-1}} \in \mathbb{F}_q$, for all $a \in \mathbb{F}_{q^k}$, since $N(a)^q = N(a)$.

Let $\mathbb{F}$ denote an arbitrary field. We denote by $\mathbb{P}_n(\mathbb{F})$ the projective space arising from the $(n+1)$-dimensional vector space over $\mathbb{F}$. Throughout dim will refer to projective dimension. A point of $\mathbb{P}_n(\mathbb{F})$ (which is a one-dimensional subspace of the vector space) will often be written as $\langle u \rangle$, where u is a vector in the $(n+1)$-dimensional vector space over $\mathbb{F}$.

Let $\Gamma(t)$ be the graph with vertices $(a, \alpha) \in \mathbb{F}_{q^{t-1}} \times \mathbb{F}_q$, $\alpha \neq 0$, where (a, α) is joined to (a', α') if and only if $N(a + a') = \alpha\alpha'$. The graph $\Gamma(t)$ was constructed in [11], where it was shown to contain no copy of $K_{t,t!+1}$. In [1] Alon, Rónyai and Szabó proved that $\Gamma(t)$ contains no copy of $K_{t,(t-1)!+1}$. Our aim here is to show that it also contains no $K_{t+1,(t-1)!-1}$, generalizing the same result for $t = 5$ presented in [2].

Let

$$V = \{(1, a) \otimes (1, a^q) \otimes \cdots \otimes (1, a^{q^{t-2}}) \mid a \in \mathbb{F}_{q^{t-1}}\} \subset \mathbb{P}_{2^{t-1}-1}(\mathbb{F}_{q^{t-1}}).$$

The set V is the affine part of an algebraic variety that is in turn a subvariety of the Segre variety

$$\Sigma = \underbrace{\mathbb{P}_1 \times \mathbb{P}_1 \times \cdots \times \mathbb{P}_1}_{t-1 \text{ times}},$$

where $\mathbb{P}_1 = \mathbb{P}_1(\mathbb{F}_q)$.

The affine point $(1, a) \otimes (1, a^q) \otimes \cdots \otimes (1, a^{q^{t-2}})$ has coordinates indexed by the subsets of $T := \{0, 1, \ldots, t-1\}$, where the S-coordinate is

$$(\prod_{i \in S} a^{q^i}),$$

for any non-empty subset S of T and

$$\prod_{i \in S} a^{q^i} = 1$$

when $S = \emptyset$ (see [13]).

Let $n = 2^{t-1} - 1$.

We order the coordinates of $\mathbb{P}_n(\mathbb{F}_{q^{t-1}})$ so that if the i-th coordinate corresponds to the subset S, then the $(n-i)$–th coordinate corresponds to the subset $T \setminus S$.

Embed the $\mathbb{P}_n(\mathbb{F}_{q^{t-1}})$ containing V as a hyperplane section of $\mathbb{P}_{n+1}(\mathbb{F}_{q^{t-1}})$ defined by the equation $x_{n+1} = 0$.

Let β be the symmetric bilinear form on the $(n+2)$-dimensional vector space over $\mathbb{F}_{q^{t-1}}$ defined by

$$\beta(u, v) = \sum_{i=0}^{n} u_i v_{n-i} - u_{n+1} v_{n+1}.$$

Let $\perp$ be defined in the usual way, so that given a subspace Π of $\mathbb{P}_{n+1}(\mathbb{F}_{q^{t-1}})$, $\Pi^\perp$ is the subspace of $\mathbb{P}_{n+1}(\mathbb{F}_{q^{t-1}})$ defined by

$$\Pi^\perp = \{v \mid \beta(u, v) = 0, \text{ for all } u \in \Pi\}.$$

We wish to define the same graph $\Gamma(t)$, so that adjacency is given by the bilinear form. Let $P_\infty = (0, 0, 0, \ldots, 1)$. Let Γ' be a graph with vertex set the set of points on the lines joining the points of V to P_∞ obtained using only scalars in $\mathbb{F}_q$, distinct from P_∞ and not contained in the hyperplane $x_{n+1} = 0$. Join two vertices $\langle u \rangle$ and $\langle u' \rangle$ in Γ' if and only if $\beta(u, u') = 0$. It is a simple matter to verify that the graph Γ' is isomorphic to the graph $\Gamma(t)$ since

$$N(a + b) = \sum_{S \subseteq T} \prod_{i \in S,\ j \in T \setminus S} a^{q^i} b^{q^j} = \beta(u, v) + u_{n+1} v_{n+1},$$

where

$$u = (1, a) \otimes (1, a^q) \otimes \cdots \otimes (1, a^{q^{t-2}}),$$

and

$$v = (1, b) \otimes (1, b^q) \otimes \cdots \otimes (1, b^{q^{t-2}}).$$

We shall refer to Γ' as $\Gamma(t)$ from now on.

We recall some known properties of Σ and its subvariety

$$\mathcal{V} = \{(a, b) \otimes (a^q, b^q) \otimes \cdots (a^{q^{t-2}}, b^{q^{t-2}}) \mid (a, b) \in \mathbb{P}_1(\mathbb{F}_{q^{t-1}})\}$$

and prove a new one in Theorem 2.5.

Let $\overline{\mathbb{F}_q}$ denote the algebraic closure of $\mathbb{F}_q$ and consider Σ as the Segre variety over $\overline{\mathbb{F}_q}$.

Theorem 2.1. *Σ is a smooth irreducible variety.*

Theorem 2.2. *The dimension of Σ (as algebraic variety) is $t-1$ and its degree is $(t-1)!$.*

Theorem 2.3. [13] *Any t points of $\mathcal{V}$ are in general position.*

Theorem 2.4. [10] *If $t+1$ points span a $(t-1)$-dimensional projective space, then that space contains $q+1$ points of $\mathcal{V}$.*

THEOREM 2.5. *If a subspace of codimension t contains a finite number of points of Σ then it contains at most $(t-1)!-2$ points of Σ.*

Proof. By Theorem 2.1, Σ is smooth, so it is regular at each of its points, i.e., if $T_P\Sigma$ is the tangent space of Σ at a point $P \in \Sigma$, then $\dim T_P\Sigma = t-1$.

Let Π be a subspace of codimension t containing a finite number of points of Σ. Let $P \in \Pi \cap \Sigma$. Then $\dim\langle T_P\Sigma, \Pi\rangle \leqslant n-1$. Therefore, there is a hyperplane H containing $\langle T_P\Sigma, \Pi\rangle$.

Suppose that H contains another tangent space $T_R\Sigma$, with $R \in \Pi \cap \Sigma$. The algebraic variety $H \cap \Sigma$ has dimension $t-2$ (since Σ is irreducible) and it has two singular points, P and R. Since $\dim H \cap \Sigma = t-2$ as an algebraic variety, there must be a linear subspace Π_1 of codimension $t-2$ in H containing Π and such that $\Pi_1 \cap H \cap \Sigma$ consists of $\deg(H \cap \Sigma) \leqslant (t-1)!$ points of Σ counted with their multiplicity. Since Π_1 contains P and R, which are singular points and so with multiplicity at least 2, we have that

$$|\Pi \cap \Sigma| \leqslant |\Pi_1 \cap \Sigma| \leqslant (t-1)! - 2.$$

Suppose now that H does not contain any other tangent space $T_R\Sigma$ with $R \in \Pi \cap \Sigma$, $R \neq P$. Then take $R \in \Pi \cap \Sigma$ and consider a hyperplane $H' \neq H$ containing $\langle T_R\Sigma, \Pi\rangle$. Then the tangent spaces of P and R with respect to $H \cap H' \cap \Sigma$ are $T_P\Sigma \cap H'$ and $T_R\Sigma \cap H$, and they both have dimension $t-2$ (as linear spaces).

If $\dim H \cap H' \cap \Sigma = t-3$ as an algebraic variety, then P and R are two singular points of $H \cap H' \cap \Sigma$ and we can find, as before, a linear subspace Π_1 of codimension $t-3$ in $H \cap H'$ such that it contains Π and intersects $H \cap H' \cap \Sigma$ in $\deg(H \cap H' \cap \Sigma) \leqslant (t-1)!$ points, counted with their multiplicity. Since P and R have multiplicity at least 2, we have

$$|\Pi \cap \Sigma| \leqslant |\Pi_1 \cap \Sigma| \leqslant (t-1)! - 2.$$

If $\dim H \cap H' \cap \Sigma = t-2$ as an algebraic variety, then $H \cap \Sigma$ is reducible. Hence, we have

$$H \cap \Sigma = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_r,$$

where $\mathcal{V}_i$ is an irreducible variety of dimension $t-2$, for all $i = 1, \ldots, r$. So we have

$$H \cap H' \cap \Sigma = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_s \cup \mathcal{W}_{s+1} \cup \mathcal{W}_{s+2} \cup \cdots \cup \mathcal{W}_r,$$

where $\mathcal{W}_i$ is a hyperplane section of $\mathcal{V}_i$, for all $i = s+1, \ldots, r$. We observe that also $H' \cap \Sigma$ has to be reducible and, since the decomposition in irreducible components is unique, we have

$$H' \cap \Sigma = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_s \cup \mathcal{V}'_{s+1} \cup \mathcal{V}'_{s+2} \cup \cdots \cup \mathcal{V}'_r,$$

where $\mathcal{V}_i$ and $\mathcal{V}'_j$ are irreducible varieties of dimension $t-2$.

We have, by hypothesis, that $T_P\Sigma \subset H$ and $P \in \Pi$. So either $P \in \mathcal{V}_i$ and it is singular for $\mathcal{V}_i$, for some $i \in \{1, 2, \ldots, r\}$, or it is not singular for $\mathcal{V}_\ell$, for any $\ell \in \{1, 2, \ldots, r\}$.

Suppose we are in the first case. We know that $P \in \Pi \subset H'$. If $\mathcal{V}_i \subseteq H'$, then P is singular for an irreducible component of $H' \cap \Sigma$ and so $T_P\Sigma \subset H'$, contradicting our hypothesis, so $\mathcal{V}_i$ is not contained in H' and $H' \cap \mathcal{V}_i = \mathcal{W}_i$. We have that $\dim T_P\Sigma \cap H' = t-2$ (as linear subspace) and $\dim \mathcal{W}_i = t-3$ (as algebraic variety), so P is singular for $\mathcal{W}_i$.

Suppose now that P is not singular for any $\mathcal{V}_i$, so the dimension of $T_P\mathcal{V}_i$, as a subspace, is $t-2$. If $P \notin \mathcal{V}_j$, for any $i \neq j$, then

$$T_P(H \cap \Sigma) = T_P(\mathcal{V}_i) = T_P(\Sigma),$$

a contradiction since the dimension of $T_P(\Sigma)$ is $t-1$. Hence $P \in \mathcal{V}_i \cap \mathcal{V}_j$, and so P is contained in the intersection of two components of $H' \cap \Sigma$, so it is again a singular (or multiple) point. The same is true for the point R such that $T_R\Sigma \subset H'$, so in

$$\mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_s \cup \mathcal{W}_{s+1} \cup \mathcal{W}_{s+2} \cup \cdots \cup \mathcal{W}_r$$

there are at least two multiple points and when we sum up all the degrees, we count at least two points twice, hence, by

$$\sum_{i=1}^{s} \deg \mathcal{V}_i + \sum_{j=s+1}^{r} \deg \mathcal{W}_j \leqslant (t-1)!,$$

we get that the number of points in

$$\Pi \cap (\mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_s \cup \mathcal{W}_{s+1} \cup \mathcal{W}_{s+2} \cup \cdots \cup \mathcal{W}_r),$$

is at most $(t-1)! - 2$. □

Remark One could wonder whether one could try with one more hyperplane H'' such that $T_Q\Sigma \subset H''$, $T_Q\Sigma \not\subseteq H$,$T_Q\Sigma \not\subseteq H$' and $Q \in \Pi$. However, it can happen that $H \cap H' \cap H'' = H \cap H'$, so $\dim T_Q\Sigma \cap H \cap H' \cap H'' = t-2$ (as linear space) and $\dim H \cap H' \cap H'' \cap \Sigma = t-2$, so Q would not be a singular point of

$$H \cap H' \cap H'' \cap \Sigma = H \cap H' \cap \Sigma.$$

THEOREM 2.6. *For $q \geqslant (t-1)! + 1$ the graph $\Gamma(t)$ contains no $K_{t+1,(t-1)!-1}$.*

Proof. Let $X = \{x_1, x_2, \ldots, x_{t+1}\}$ be $t+1$ distinct vertices of $\Gamma(t)$. The set of common neighbours of the elements of X is $\Pi^\perp \cap \Gamma(t)$, where Π is the subspace spanned by X. If any two elements of X project from P_∞ onto the same point of V, then $P_\infty \in \Pi$ and hence $\Pi^\perp \subset P_\infty^\perp$. Since $P_\infty^\perp$ is the hyperplane $x_{n+1} = 0$, $\Pi^\perp \cap \Gamma(t) = \emptyset$, and the elements of X have no common neighbour.

Therefore, we assume now that all the points in X project from P_∞ onto distinct points of V. Then, by Theorem 2.3, $\dim \Pi \geqslant t-1$.

If $\dim \Pi = t-1$, then by Theorem 2.3, the projection of Π onto V contains at least q points of V. Therefore, there are at least q points Y of Π on the lines joining P_∞ to the points of V. We wish to prove that the points of Y are vertices of the graph $\Gamma(t)$. To do this, we have to show that the points of Y, which are of the form $\langle (v, \lambda) \rangle$, where

$v \in V$ and $\lambda \in \overline{\mathbb{F}_q}$, are of the form $\langle(v,\lambda)\rangle$, where $v \in V$ and $\lambda \in \mathbb{F}_q$. Assuming that the vertices in X have at least two common neighbours, we can suppose that there is a common neighbour of the elements of X of the form $\langle(u,\mu)\rangle$, where $u \in V$, $u \neq -v$ and $\mu \in \mathbb{F}_q$, is a common neighbour of the elements of X. Then $\langle(u,\mu)\rangle$ is in $\Pi^\perp$ and since $Y \subset \Pi$,

$$N(u+v) = \lambda\mu.$$

Since $N(u+v) \in \mathbb{F}_q$ and $\mu \in \mathbb{F}_q$, we have that $\lambda \in \mathbb{F}_q$ and so the points of Y are vertices of the graph $\Gamma(t)$. Therefore, the vertices of X have at least q common neighbours. Since Γ contains no $K_{t,(t-1)!+1}$, if $q \geqslant (t-1)!+1$, then this case cannot occur.

If $\dim \Pi = t$ then $\dim \Pi^\perp = n-t$. Let Y be the points of $\Pi^\perp$ which project from P_∞ onto V. Arguing as in the previous paragraph, the points Y are vertices of the graph $\Gamma(t)$. Since the vertices of X have at most $(t-1)!+1$ common neighbours, there are a finite number of points in Y and so a finite number of points in the projection of $\Pi^\perp$ onto V. By Theorem 2.5, this projection contains at most $(t-1)!-2$ points of V, so there are at most $(t-1)!-2$ points in Y. Therefore, the vertices in X have at most $(t-1)!-2$ common neighbours. □

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Simeon Ball

Departament de Matemàtica Aplicada IV,

Universitat Politècnica de Catalunya, Jordi Girona 1-3, Mòdul C3, Campus Nord,

08034 Barcelona, Spain
simeon@ma4.upc.edu

Valentina Pepe
SBAI Department,
Sapienza University of Rome, Via Antonio Scarpa 16
00161 Rome, Italy
valepepe@sbai.uniroma1.it